A few remarks on integral representation for zonal spherical functions on the symmetric space $SU(N)/SO(N,\mathbb{R})$.

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Abstract

The integral representation on the orthogonal groups for zonal spherical functions on the symmetric space $SU(N)/SO(N,\mathbb{R})$ is used to obtain a generating function for such functions. For the case $N = 3$ the three–dimensional integral representation reduces to a one–dimensional one.

1 Introduction

The interest of studying classical and quantum integrable systems is always increasing. These systems present some very nice characteristics which are related to different algebraic and analytic properties. For instance, the connection of completely integrable classical Hamiltonian systems with semisimple Lie algebras was established more than twenty years ago in [OP 1976] and the relationship with quantum systems in [OP 1977].

On the other side, it was also shown in [OP 1978] and [OP 1983] the possibility of finding the explicit form of the Laplace–Beltrami operator for each

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symmetric space appearing in the classification given in the classical Helgason's book [He 1978] by associating to it a quantum mechanical problem.

The search for the eigenfunctions of such operators is not an easy task. These functions are but the so–called zonal spherical functions and for one special case and for the case of symmetric spaces with root systems of the type A_{N-1} were found explicitly in [Pr 1984].

Our aim in this letter is to present some remarks concerning the integral representation for zonal spherical functions on the symmetric space $SU(N)/SO(N,\mathbb{R})$. This representation will be used for obtaining a generating function for such zonal spherical functions.

We recall that if G is a connected real semisimple Lie group and T^ρ denotes an irreducible unitary representation of G with support in the Hilbert space $\mathcal{H}$, where ρ is a parameter characterizing the representation, the representation T^ρ is said to be of class I if there exists a vector $|\Psi_0\rangle$ such that $T^\rho(k)|\Psi_0\rangle = |\Psi_0\rangle$, for any element k in the maximal compact subgroup K of G. The function defined by the expectation value of T^ρ is called a zonal spherical function belonging to the representation T^ρ. Zonal spherical functions satisfy a kind of completeness condition like that of coherent states.

The paper is organized as follows. In order to the paper to be more self-contained we give in Section 2 the general definitions and properties on zonal spherical functions. The particular case $N = 2$ is considered in Section 3, and then the formulae are extended in Section 4 to the case $N = 3$. Section 5 is devoted to introduce an integral representation for the generating function for zonal spherical functions for the symmetric space $SU(N)/SO(N,\mathbb{R})$ and the integrals arising in the expresion are explicitly computed in the particular cases $N = 2$ and $N = 3$.

2 Zonal spherical functions

Let $G^- = SL(N,\mathbb{R})$ be the group of real matrices of order N with determinant equal to one. This group contains three important subgroups, to be denoted K, A and $\mathcal{N}$. The subgroup $K = SO(N,\mathbb{R})$ is the compact group of real orthogonal matrices, the subgroup A is the Abelian group of inversible real diagonal matrices and $\mathcal{N}$ is the subgroup of lower triangular real matrices with units on the principal diagonal, which is a nilpotent group.

Using the polar decomposition of a matrix, the homogeneous space $X^- =$

G^-/K can be identified with the space of real positive–definite symmetric matrices with determinant equal to one. It is known that any element $g \in G^-$ may be decomposed in a unique way as a product $g = kan$, with $k \in K$, $a \in A$ and $n \in \mathcal{N}$, respectively, so-called Iwasawa decomposition. We denote the elements in such a factorization as $k(g)$, $a(g)$ and $n(g)$, i.e. $g = k(g)\,a(g)\,n(g)$. Correspondingly, the linear space underlying the Lie algebra $\mathfrak{g}$ of G^- can be decomposed as a direct sum of the linear spaces of the Lie subalgebras $\mathfrak{k}$ of K, $\mathfrak{a}$ of A and $\mathfrak{n}$ of $\mathcal{N}$, i.e., $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let us also denote as $\mathfrak{a}^*$ the dual space of $\mathfrak{a}$ and so on.

There are natural left and right actions of group G^- on K and $\mathcal{N}$, respectively, induced by left and right multiplication, respectively, which are defined by the formulae

$$k^g = k(gk), \quad n_g = n(ng), \tag{2.1}$$

and for any $\lambda \in \mathfrak{a}^*$, we may construct the representation $T^\lambda(g)$ of the group G^- in the space of $L^2(K)$ or $L^2(\mathcal{N})$ of square integrable functions on K or $\mathcal{N}$ by the formula

$$[T^\lambda(g)\,f](k) = \exp\,((i\lambda - \rho,\, H(gk))\,f(k^{g^{-1}}), \tag{2.2}$$

or

$$[T^\lambda(g)\,f](n) = \exp\,((i\lambda - \rho,\, H(ng))\,f(n_g), \tag{2.3}$$

where $H(g)$ is defined by $a(g) = \exp\,H(g)$ and ρ is given by one half of the sum of positive roots of the symmetric space X^-,

$$\rho = \frac{1}{2} \sum_{R^+} \alpha.$$

This so called representation of principal series is unitary and irreducible. It has the property that in the Hilbert space $\mathcal{H}^\lambda$ there is a normalized vector $|\Psi_0\rangle \in \mathcal{H}^\lambda$ which is invariant under the action of group K:

$$T^\lambda(k)\,|\Psi_0\rangle = |\Psi_0\rangle, \tag{2.4}$$

Let us consider the function

$$\Phi_\lambda(g) = \langle\Psi_0|\,T^\lambda(g)\,|\Psi_0\rangle. \tag{2.5}$$

This function is called a zonal spherical function and has the properties of

$$\Phi_\lambda(k_1 g k_2) = \Phi_\lambda(g),\ \Phi_\lambda(k) = 1,\ \forall k \in K,\ \Phi_\lambda(e) = 1. \tag{2.6}$$

For the realization of $\mathcal{H}^\lambda$ as $L^2(K)$, we take $|\Psi_0\rangle$ as the constant function $\Psi_0(k) \equiv 1$, and then we have the integral representation for $\Phi_\lambda(g)$:

$$\Phi_\lambda(g) = \int_K \exp\ ((i\lambda - \rho,\ H(gk))\, d\mu(k), \quad \int_K d\mu(k) = 1, \tag{2.7}$$

where $d\mu(k)$ denotes an invariant (under G^-) measure on K. Note that due to (2.6) the function $\Phi_\lambda(g)$ is completely defined by the values $\Phi_\lambda(a)$, $a \in A$.

Here $\Phi_\lambda(g)$ is the eigenfunction of Laplace-Beltrami Δ_j operators and correspondingly $\Phi_\lambda(a)$ is the eigenfunction of radial parts Δ_j^0 of these operators, in particular,

$$\Delta_2^0 = \sum_{j=1}^N \partial_j^2 + 2\kappa \sum_{j<k}^N \coth(q_j - q_k)(\partial_j - \partial_k),\ \kappa = \frac{1}{2},\ \partial_j = \frac{\partial}{\partial q_j},\ a_j = e^{q_j}. \tag{2.8}$$

Note that the analogous consideration of groups $G^- = SL(N, \mathbb{C})$ and $G^- = SL(N, \mathbb{H})$ over complex numbers and quaternions gives us the corresponding integral representations for $\kappa = 1$ and $\kappa = 2$.

Note that the above construction is also valid for the dual spaces $X^+ = G^+/K$, where $G^+ = SU(N)$ is the group of unitary matrices with determinant equal to one. In this case the representation $T^\lambda(g)$ is defined by a set $l = (l_1, \ldots, l_{N-1})$ of $(N-1)$ nonnegative integer numbers l_j and the integral representation (2.7) takes the form

$$\Phi_l(g) = \int_K \exp(l, H(gk))\, d\mu(k), \quad \int_K d\mu(k) = 1, \tag{2.9}$$

and $\Phi_l(g)$ is the eigenfunction of the radial part of the Laplace–Beltrami operator

$$\Delta_2^0 = \sum_{j=1}^N \partial_j^2 + 2\kappa \sum_{j<k}^N \mathrm{cotg}(q_j - q_k)(\partial_j - \partial_k),\ \kappa = \frac{1}{2},\ \partial_j = \frac{\partial}{\partial q_j},\ a_j = x_j = e^{iq_j}. \tag{2.10}$$

The element k of the group $SO(N,\mathbb{R})$ is the matrix (k_{ij}) and may be considered as the set of N unit orthogonal vectors $k^{(j)} = (k_{1j}, \ldots, k_{Nj})$ from which we may construct the set of polyvectors

$$k^{(i)},\ k^{(i_1,i_2)} = k^{(i_1)} \wedge k^{(i_2)},\ k^{(i_1,i_2,i_3)} = k^{(i_1)} \wedge k^{(i_2)} \wedge k^{(i_3)},\ \cdots . \qquad (2.11)$$

There is a natural action of the group G on the space of polyvectors and the integral representation (2.9) may be written now in the form

$$\Phi_l(x_1,\ldots,x_N) = \int \Xi_1^{l_1}(x;k) \cdots \Xi_{N-1}^{l_{N-1}}(x;k)\, d\mu(k^{(1)},\ldots,k^{(N-1)}), \qquad (2.12)$$

where

$$\Xi_1(x;k) = \sum_j k_j^{(1)\,2} x_j,\ \Xi_2(x;k) = \sum_{i<j} (k^{(1)} \wedge k^{(2)})_{ij}^2\, x_i x_j,$$
$$\Xi_3(x;k) = \sum_{i<j<l} (k^{(1)} \wedge k^{(2)} \wedge k^{(3)})_{ijl}^2\, x_i x_j x_l, \ldots .$$

Here $d\mu(k^{(1)},\ldots,k^{(N-1)})$ is the invariant measure on K such that

$$\int_K d\mu(k^{(1)},\ldots,k^{(N-1)}) = 1. \qquad (2.13)$$

3 The case $N = 2$

In this case, the integral representation takes the form

$$\Phi_l(x_1,x_2) = \int [(k'ak)_{11}]^l\, d\mu(k) = \int (k_{11}^2 x_1 + k_{21}^2 x_2)^l\, d\mu(k), \quad \int d\mu(k) = 1, \qquad (3.1)$$

where k' is the transpose matrix of k, or

$$\Phi_l(x_1,x_2) = \int_{S^1} (n_1^2 x_1 + n_2^2 x_2)^l\, d\mu(n), \quad (n,n) = n_1^2 + n_2^2 = 1, \qquad (3.2)$$

where $d\mu(n) = \frac{1}{2\pi}\, d\varphi$ is an invariant measure on an unit circle S^1 in $\mathbb{R}^2$.

So,

$$\Phi_l(x_1,x_2) = \sum_{k_1+k_2=l} C^l_{k_1,k_2} x_1^{k_1} x_2^{k_2}, \qquad (3.3)$$

and

$$C^l_{k_1,k_2} = \frac{l!}{k_1!k_2!}\,\langle n_1^{2k_1} n_2^{2k_2}\rangle, \quad \langle n_1^{2k_1} n_2^{2k_2}\rangle = \int_{S^1} n_1^{2k_1} n_2^{2k_2}\, d\mu(n). \tag{3.4}$$

The integral is easily calculated by using a standard parametrization $n_1 = \cos\varphi,\ n_2 = \sin\varphi,\ d\mu(n) = \frac{1}{2\pi}\,d\varphi$. We obtain

$$\langle n_1^{2k_1} n_2^{2k_2}\rangle = \frac{(\frac{1}{2})_{k_1}\,(\frac{1}{2})_{k_2}}{(1)_{k_1+k_2}}, \tag{3.5}$$

where $(a)_k$ is the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$. So finally we have

$$C^l_{k_1k_2} = \frac{(\frac{1}{2})_{k_1}\,(\frac{1}{2})_{k_2}}{(1)_{k_1}\,(1)_{k_2}}, \quad l = k_1 + k_2, \tag{3.6}$$

$$\Phi_l(x_1,x_2) = \sum_{k_1+k_2=l} \frac{(\frac{1}{2})_{k_1}\,(\frac{1}{2})_{k_2}}{(1)_{k_1}\,(1)_{k_2}}\, x_1^{k_1} x_2^{k_2}. \tag{3.7}$$

If we put $x_1 = e^{i\theta},\ x_2 = e^{-i\theta}$, then $\Phi_l(x_1,x_2) = A_l P_l\,(\cos\theta)$, where $P_l(\cos x)$ is the Legendre polynomial.

These formulae may be easily extended to the N-dimensional case. Namely, we have

$$\Phi_{(l,0,\ldots,0)}(x_1,\ldots,x_N) = \int_{S^{N-1}} (n_1^2 x_1 + \ldots + n_N^2 x_N)^l\, d\mu(n), \quad \int d\mu(n) = 1, \tag{3.8}$$

where $d\mu(n)$ is invariant measure on S^{N-1} and

$$\begin{aligned} \Phi_{(l,0,\ldots,0)}(x_1,\ldots,x_N) &= \sum_{k_1+\ldots+k_N=l} C^l_{k_1\ldots k_N}\, x_1^{k_1}\ldots x_N^{k_N}, \\ C^l_{k_1,\ldots k_N} &= \frac{l!}{k_1!\ldots k_N!}\,\langle n_1^{2k_1}\ldots n_N^{2k_N}\rangle, \\ \langle n_1^{2k_1}\ldots n_N^{2k_N}\rangle &= \frac{(\frac{1}{2})_{k_1}\cdots(\frac{1}{2})_{k_N}}{(\frac{N}{2})_l}. \end{aligned} \tag{3.9}$$

So

$$C^l_{k_1\ldots k_N} = \frac{(\frac{1}{2})_{k_1}\cdots(\frac{1}{2})_{k_N}}{(1)_{k_1}\,\ldots(1)_{k_N}}\,\frac{(1)_l}{(\frac{N}{2})_l}, \quad l = k_1 + \ldots + k_N. \tag{3.10}$$

4 The case $N = 3$

In this case, the element of the orthogonal group $SO(3, \mathbb{R})$ has the form

$$\mathrm{k} = \begin{pmatrix} n_1 & l_1 & m_1 \\ n_2 & l_2 & m_2 \\ n_3 & l_3 & m_3 \end{pmatrix},$$

i.e., it may be represented by the three unit orthogonal each other vectors

$$n,\ l,\ m;\ n^2 = l^2 = m^2 = 1,\ (n, l) = (l, m) = (m, n) = 0,$$

and the integral representation for zonal spherical polynomials takes the form

$$\Phi_{pq}(x) = \int_K (n_1^2 x_1 + n_2^2 x_2 + n_3^2 x_3)^p \left(\sum_{j<k} (n_j l_k - n_k l_j)^2 \, x_j x_k \right)^q d\mu(n, l), \quad (4.1)$$

where the integration is taken on the orthogonal group $K = SO(3, \mathbb{R})$, what is equivalent to the space of two unit orthogonal vectors n and l.

Note that $m_k = \epsilon_{kij} n_i l_j$; we also have $x_1 x_2 = x_3^{-1}, \ldots$. Hence,

$$\Phi_{pq}(x_1, x_2, x_3) = \int_K (n_1^2 x_1 + n_2^2 x_2 + n_3^2 x_3)^p \left(m_1^2 x_1^{-1} + m_2^2 x_2^{-1} + m_3^2 x_3^{-1} \right)^q \, d\mu(n, m). \quad (4.2)$$

For vectors n and m the standard parametrization through Euler angles φ, θ and ψ, may be used:

$$\begin{aligned} n &= (\cos\varphi \ \sin\theta, \sin\varphi \ \sin\theta, \cos\theta), \quad m = \cos\psi \cdot a + \sin\psi \cdot b, \\ a &= (-\sin\varphi, \cos\varphi, 0), \ b = (-\cos\varphi \ \cos\theta, -\sin\varphi \ \cos\theta, \sin\theta) \end{aligned} \quad (4.3)$$

with $d\mu(k) = d\mu(n, m) = A \sin\theta \, d\theta \, d\varphi \, d\psi$, and in the preceding expression we have a three–dimensional integral which may be calculated using the generating functions.

5 Generating functions

Let us define the generating function by the formula

$$F(x_1, x_2, \ldots, x_N; t_1, \ldots, t_{N-1}) = \sum \Phi_{l_1 \cdots l_{N-1}}(x_1, \ldots, x_N) \, t_1^{l_1} \cdots t_{N-1}^{l_{N-1}}. \quad (5.1)$$

Then we have the integral representation

$$F(x_1, x_2, \ldots, x_N; t_1, \ldots, t_{N-1}) = \int [\prod_{j=1}^{N-1} (1 - \Xi_j(x; k) t_j)]^{-1} \, d\mu(k). \qquad (5.2)$$

Let us introduce the coordinate system such that a and b are two unit orthogonal vectors in the two–dimensional plane orthogonal to the set of vectors $\{k^{(1)}, \ldots, k^{(N-2)}\}$. Then, an arbitrary unit vector n in this plane has the form $\cos\psi \cdot a + \sin\psi \cdot b$, and we may integrate first on $d\mu(n)$. The integral representation (5.2) takes the form:

$$F(x_1, x_2, \ldots, x_N; t_1, \ldots, t_{N-1}) = \int [A_{ij} n_i n_j]^{-1} \, d\mu^{(N-2)}(k) \, d\mu(n). \qquad (5.3)$$

The integral on $d\mu(n)$ may be easily calculated and we have

$$F(x_1, x_2, \ldots, x_N; t_1, \ldots, t_{N-1}) = \int [D]^{-1/2} \, d\mu(k^{(1)}, \ldots, k^{(N-2)}), \qquad (5.4)$$

where $D = \det(A_{ij}),\ A_{ij} = A_{ij}(x; k^{(1)}, \ldots, k^{(N-2)})$.

In the simplest case $N = 2$, we have

$$F(x_1, x_2; t) = [(1 - x_1 t)(1 - x_2 t)]^{-1/2}, \qquad (5.5)$$

from which the formula (3.7) for $\Phi_l(x_1, x_2)$ follows.

In the case $N = 3$, the integration on $d\mu(\psi)$ gives

$$F(x_1, x_2, x_3; t_1, t_2) = \int B^{-1}(n) C^{-1/2}(n) \, d\mu(n), \quad \int d\mu(n) = 1, \qquad (5.6)$$

where

$$B = 1 - (n_1^2 x_1 + n_2^2 x_2 + n_3^2 x_3) t_1, \quad C = (1 - x_2^{-1} t_2)(1 - x_3^{-1} t_2) n_1^2 + \cdots \qquad (5.7)$$

The crucial step for further integration is the use of the formula

$$B^{-1} C^{-1/2} = \int_0^1 d\xi \, [B(1 - \xi^2) + C\xi^2]^{-3/2}. \qquad (5.8)$$

Using this formula we obtain

$$F(x_1,x_2,x_3;t_1,t_2)=\int_0^1 d\xi\,\int [E(x_1,x_2,x_3;t_1,t_2,n,\xi)]^{-3/2}\,d\mu(n), \qquad (5.9)$$

where

$$E(x_1,x_2,x_3;t_1,t_2,n,\xi)=\sum_j e_j(x_1,x_2,x_3;t_1,t_2,\xi)n_j^2. \qquad (5.10)$$

We can now integrate on $d\mu(n)$ and finally we obtain the one–dimensional integral representation for the generating function

$$F(x_1,x_2,x_3;t_1,t_2)=\int_0^1 d\xi\,[H(x_1,x_2,x_3;t_1,t_2,\xi)]^{-1/2}, \qquad (5.11)$$

where $H=e_1e_2e_3$ and the functions $e_j(\xi;t_1,t_2)$ are given by

$$h_j(\xi;t_1,t_2)=1-d_j(t_1,t_2)(1-\xi^2), \quad d_j(t_1,t_2)=(x_jt_1+x_j^{-1}t_2-t_1t_2). \qquad (5.12)$$

From this it follows that if $z_1=x_1+x_2+x_3$, and $z_2=x_1x_2+x_2x_3+x_3x_1$, then

$$\begin{aligned} H &= a_0^3-a_0^2[z_1\tau_1+z_2\tau_2]+a_0[z_2\tau_1^2+z_1\tau_2^2+(z_1z_2-3)\tau_1\tau_2] \\ &- [\tau_1^3+\tau_2^3+\tau_1\tau_2[(z_2^2-2z_1)\tau_1+(z_1^2-2z_2)\tau_2]] \end{aligned} \qquad (5.13)$$

where $a_0=1+(1-\xi^2)t_1t_2,\ \tau_1=(1-\xi^2)t_1,\ \tau_2=(1-\xi^2)t_2$. Note that from (5.13) it follows that the integral (5.11) is elliptic and it may be expressed in terms of standard elliptic integrals.

Expanding $F(x_1,x_2,x_3;t_1,t_2)$ in power series of the variable t_2 one obtains

$$F(x_1,x_2,x_3;t_1,t_2)=\sum_{q=0}^{\infty} F_q(x_1,x_2,x_3;t_1)\,t_2^q \qquad (5.14)$$

and we have

$$F_0(x_1,x_2,x_3;t)=\int_0^1 d\xi\,[H_0]^{-1/2}, \qquad (5.15)$$

and

$$F_1=\frac{1}{2}\int_0^1 d\xi\,H_1[H_0]^{-3/2} \qquad (5.16)$$

where

$$
\begin{aligned}
H_0 &= 1 - z_1\tau_1 + z_2\tau_1^2 - \tau_1^3, \\
H_1 &= (1-\xi^2)z_2 - [3\xi^2 + z_1 z_2(1-\xi^2)]\tau_1 + [2z_1\xi^2 + (1-\xi^2)z_2^2]\tau_1^2 - z_2\tau_1^3.
\end{aligned}
$$

From the integral representation (5.11) many useful formulae may be obtained, here we give just one of them: when z_1 and z_2 go to infinity,

$$
\Phi_{pq}(z_1, z_2) \approx A_{pq}\, z_1^p z_2^q, \quad A_{pq} = \frac{(\frac{1}{2})_p(\frac{1}{2})_q}{(1)_p(1)_q}\,\frac{(1)_{p+q}}{(\frac{3}{2})_{p+q}} \tag{5.17}
$$

A more detailed version of this note will be published elsewhere.

Acknowledgements. One of the authors (A.P.) would like to thank the Department of Theoretical Physics of Zaragoza University for its hospitality. Financial support from Dirección General de Enseñanza Superior, Spain (SAB95-0447) is also acknowledged.

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